§1.2 The superconformal algebra

In conformal field theory the $(d-1,1)$ Lorentzian spinor $Q_\alpha \longrightarrow (d,2)$ conformal spinor

Let us choose Γ matrices for $SO(d,2)$:

$$\Gamma_\mu = \begin{pmatrix} \sigma_\mu & 0 \\ 0 & -\sigma_\mu \end{pmatrix}$$

$$\Gamma_{-1} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$\Gamma_d = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

where σ_μ are $SO(d-1,1)$ Γ matrices.

Γ_μ are constructed iteratively from the $d=2$ expressions:

$$\sigma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then the above Γ matrices satisfy:

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$$

where $\eta_{ab} = \mathrm{diag}(\underset{\underset{a=-1}{\uparrow}}{-1}, \underset{\underset{a=0}{\uparrow}}{-1}, 1, \dots, 1)$

For odd d we have to add

$$\Gamma_{d+1} = \begin{pmatrix} \sigma_{d+1} & 0 \\ 0 & -\sigma_{d+1} \end{pmatrix}$$

Q is completed to full conformal spinor V through new Lorentz spinor S

$$V = \begin{pmatrix} Q_\alpha \\ C_{\theta\phi} \bar{S}^\phi \end{pmatrix}$$

where C is the charge conjugation matrix

$$C \Gamma_\mu C^{-1} = -\Gamma_\mu^T$$

$C = B\sigma_0$ where B is the matrix used to impose the Majorana condition

$$Q^\dagger = BQ$$

We set

$$[S_{ab}, V_\alpha] = R(M_{ab})_\alpha{}^\beta V_\beta$$

with $R(M_{ab}) = (i/4)[\Gamma_a, \Gamma_b]$. Specifically

$$R(P_\mu) = (-i)\begin{pmatrix} 0 & 0 \\ \sigma_\mu & 0 \end{pmatrix}$$

$$R(K_\mu) = (-i)\begin{pmatrix} 0 & \sigma_\mu \\ 0 & 0 \end{pmatrix}$$

$$R(D) = (-i/2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$R(M_{\mu\nu}) = \begin{pmatrix} R(m_{\mu\nu}) & 0 \\ 0 & R(m_{\mu\nu}) \end{pmatrix}$$

where $R(m_{\mu\nu}) = \frac{i}{4}[\sigma_\mu, \sigma_\nu]$

Euclidean spinors:

$$Q' = \frac{1}{\sqrt{2}}\left(Q - i\sigma_0 S\right) \qquad S' = \frac{1}{\sqrt{2}}\left(Q + i\sigma_0 S\right)$$

This gives

$$[M'_{pq}, Q'_\alpha] = \left(\tfrac{i}{4}\right)[\Gamma_p, \Gamma_q]_\alpha{}^\beta\, Q'_\beta$$

$$[M'_{pq}, S'_{\tilde\alpha}] = \left(\tfrac{i}{4}\right)[\tilde\Gamma_p, \tilde\Gamma_q]_{\tilde\alpha}{}^{\tilde\beta}\, Q'_{\tilde\beta}$$

$$[D', Q'_\alpha] = \left(-\tfrac{i}{2}\right) Q'_\alpha$$

$$[D', S'_{\tilde\alpha}] = \left(-\tfrac{i}{2}\right) - S'_{\tilde\alpha}$$

$$[P'_p, Q'_\alpha] = 0$$

$$[K'_p, S'_{\tilde\alpha}] = 0$$

$$[P'_p, S'_{\tilde\alpha}] = -(\tilde\Gamma_p \sigma_0)_{\tilde\alpha}{}^\beta\, Q_\beta$$

$$[K'_p, Q'_\alpha] = (\Gamma_p \sigma_0)_\alpha{}^{\tilde\beta}\, S_{\tilde\beta}$$

where $\Gamma_i = \sigma_i \;, \quad \Gamma_d = -i\sigma_0$

$\tilde\Gamma_i = \sigma_i \;, \quad \tilde\Gamma_d = i\sigma_0$

Note : σ_0 interpolates between the two reps

<u>Next:</u> need to specify the R-symmetry of the superconformal algebra

→ Jacobi-identities <u>only</u> consistent for dimensions $d = 3, 4, 5, 6$ (proof will be given later)

$\underline{d=3:}$

$SO(2,1)$ admits set of real σ matrices.
choose hermitian σ_i, anti-hermitial σ_0
We have

$$C^{-1}\sigma C = -\sigma^T$$

$$\rightarrow C = \sigma_0$$

Q and S are real

$$\Rightarrow \quad Q^+ = Q \quad ; \quad S^+ = S$$

$$Q'^+ = S' \quad ; \quad S'^+ = Q'$$

$\rightarrow$ introduce R-symmetry label i:

$$Q_\alpha \rightarrow Q_{i\alpha} \quad , \quad i=1,\dots,n$$

R-sym group: $SO(n)$
generators:

$$[I_{ij}, I_{mn}] = (-i)[I_{in}\delta_{jn} + I_{jm}\delta_{ij} - I_{im}\delta_{jn} - I_{jn}\delta_{im}]$$

$$[I_{ij}, Q_m] = (-i)[Q_i\delta_{jm} + -Q_j\delta_{im}]$$

$$[I_{ij}, Q'_m] = (-i)[Q'_i\delta_{jm} + -Q'_j\delta_{im}]$$

$$[I_{ij}, S_m] = (-i)[S_i\delta_{jm} + -S_j\delta_{im}]$$

$$[I_{ij}, S'_m] = (-i)[S'_i\delta_{jm} + -S'_j\delta_{im}]$$

$$[I_{ij}, M_{pq}] = 0$$

anti-commutation relations:

$$\{Q_{i\alpha}, Q_{j\beta}\} = (\not{P}C)_{\alpha\beta}\delta_{ij}$$

$$\{S_{i\alpha}, S_{j\beta}\} = (\cancel{K}C)_{\alpha\beta}\,\delta_{ij}$$

$$\{Q_{i\alpha}, S_{j\beta}\} = \frac{\delta_{ij}}{2}\left[(M_{\mu\nu}\Gamma_\mu\Gamma_\nu C)_{\alpha\beta} + 2DC_{\alpha\beta}\right] - C_{\alpha\beta}I_{ij}$$

Primed odd variables obey

$$\{Q'_{i\alpha}, Q'_{j\beta}\} = (\cancel{P}'C)_{\alpha\beta}\,\delta_{ij}$$

$$\{S'_{i\tilde{\alpha}}, S'_{j\tilde{\beta}}\} = (\cancel{K}'C)_{\tilde{\alpha}\tilde{\beta}}\,\delta_{ij}$$

$$\{Q'_{i\alpha}, S'_{j\tilde{\beta}}\} = i\,\frac{\delta_{ij}}{2}\left[(M'_{\mu\nu}\Gamma_\mu\Gamma_\nu C)_{\alpha\tilde{\beta}} + 2D\delta_{\alpha\tilde{\beta}}\right]$$
$$-(i)\,\delta_{\alpha\tilde{\beta}}\,I_{ij}$$

<u>$d=4$</u>:

choose Majorana spinors Q and S

$C = \sigma_0$

$$\rightarrow \quad Q^\dagger = Q \quad ; \quad S^\dagger = S$$
$$Q'^\dagger = S' \quad ; \quad S'^\dagger = Q'$$

R-symmetry: $U(n)$

Define $P_\pm = (I \pm \sigma_5)/2$

$$\rightarrow \; P_+^T = P_- \;;\; P_+^* = P_- \;;\; (P_+)^\dagger = P_+$$

generators T_{ij} of $U(n)$ obey:

$$[T_{ij}, Q_m] = [P_+ Q_i\,\delta_{jm} - P_- Q_j\,\delta_{im}]$$
$$[T_{ij}, Q'_m] = [P_+ Q'_i\,\delta_{jm} - P_- Q'_j\,\delta_{im}]$$
$$[T_{ij}, S_m] = [P_- S_i\,\delta_{jm} - P_+ S_j\,\delta_{im}]$$

$$[T_{ij}, S'_m] = [P_+ S'_i \delta_{jm} - P_- S'_j \delta_{im}]$$

$$[T_{ij}, M_{pq}] = 0$$

anti-commutation relations:

$$\{Q'_{i\alpha}, Q'_{j\beta}\} = (\not{P}'C)_{\alpha\beta}\delta_{ij}$$

$$\{S'_{i\tilde{\alpha}}, S_{j\tilde{\beta}}\} = (\not{K}'C)_{\tilde{\alpha}\tilde{\beta}}\delta_{ij}$$

$$\{Q'_{i\alpha}, S'_{j\tilde{\beta}}\} = ((i)\delta_{ij}/2)[(M'_{\mu\nu}\Gamma_\mu\Gamma_\nu)_{\alpha\tilde{\beta}} + 2\delta_{\alpha\tilde{\beta}}D']$$

$$-2(P_+)_{\alpha\tilde{\beta}}T_{ij} + 2(P_-)_{\alpha\tilde{\beta}}T_{ji} + \frac{1}{2}(\sigma_5)_{\alpha\tilde{\beta}}R$$

$\underline{d=5:}$

$SO(4,1)$ does <u>not</u> admit real spinors

$\rightarrow$ preudo real Q and S:

$$Q_{i\alpha} = \Omega_{ij}(C\sigma_0 T)_\alpha{}^\beta Q^\dagger_{j\beta}$$

$$S_{i\alpha} = \Omega_{ij}(C\sigma_0 T)_\alpha{}^\beta S^\dagger_{j\beta}$$

where Ω is $2n\times 2n$ matrix of n diagonal 2×2 blocks given by $-i\sigma_2$.

$$C T^T C^{-1} = T, \quad C^* = -C^{-1}, \quad C = -C^T$$

$$\Rightarrow \quad Q'_{i\alpha} = \Omega_{ij}(C\sigma_0 T)_\alpha{}^{\tilde{\beta}} S'^\dagger_{j\tilde{\beta}}$$

$$S'_{i\tilde{\alpha}} = \Omega_{ij}(C\sigma_0 T)_{\tilde{\alpha}}{}^\beta Q'^\dagger_{j\beta}$$

New: SCA only exists for single pair of Q,S

$\rightarrow$ R-symmetry is $Sp(1) = SU(2)$.

Denote $su(2)$ generators by T_a, $a=1,\dots,3$

$\rightarrow$ Qs and Ss transform as spinors:

$$[T_a, Q_i] = (-\sigma_a'/2)_i{}^j Q_j$$

$$[T_a, S_i] = (+\sigma_a'/2)_i{}^j S_j$$

$$[T_a, Q_i'] = (-\sigma_a'/2)_i{}^j Q_j'$$

$$[T_a, S_i'] = (-\sigma_a'/2)_i{}^j S_j'$$

odd element anti-commute:

$$\{Q'_{i\alpha}, Q'_{j\beta}\} = (\not{P}'C)_{\alpha\beta}\varepsilon_{ij}$$

$$\{S'_{i\tilde\alpha}, S_{j\tilde\beta}\} = (\not{K}'C)_{\tilde\alpha\tilde\beta}\varepsilon_{ij}$$

$$\{Q'_{i\alpha}, S'_{j\tilde\beta}\} = [(\delta_{ij}(i)/2)[(M'_{\mu\nu}\Gamma_\mu\Gamma_\nu)^\theta_\alpha + 2\delta^\theta_\alpha D']$$
$$+ 6(T_a\sigma_a'/2)_{ij}\, \delta^\theta_\alpha][(-\varepsilon_{kj}\sigma_0 C)_{\theta\tilde\beta}]$$

d=6:

SO(5,1) spinors are pseudo-real:

$$Q_{i\alpha} = \Omega_{ij}(C\sigma_0^T)_\alpha{}^\beta Q'_{j\beta}$$

$$S_{i\alpha} = \Omega_{ij}(C\sigma_0^T)_\alpha{}^\beta S^\dagger_{j\beta}$$

$$C\Gamma^T C^{-1} = -\Gamma,\quad C^T = C,\quad C^* = C^{-1}$$

Reality properties:

$$Q'_{i\alpha} = \Omega_{ij}\left((C\sigma_0{}^T\right)_\alpha{}^{\tilde\beta}\, S'^{+}_{j\tilde\beta}$$

$$S'_{i\tilde\alpha} = \Omega_{ij}(C\sigma_0{}^T)_{\tilde\alpha}{}^{\beta}\, Q'^{+}_{j\beta}$$

SCA's exist only when all Q's have same chirality $\rightarrow$ $P_+ = (1+\sigma_7)/2$

R-symmetry group is $Sp(n)$

relevant cases for physics: $n=1$, $n=2$

$Sp(1)$ $SO(5)$ $(=Sp(2))$